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On Dirac theory in the space with deformed Heisenberg algebra: exact solutions

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Abstract
The Dirac equation has been studied in which the Dirac matrices $\hat{\alpha}, \hat{\beta}$ have space factors, respectively f and f_1, dependent on the particle's space coordinates. The function f deforms Heisenberg algebra for the coordinates and momenta operators, the function f_1 being treated as a dependence of the particle mass on its position. The properties of these functions in the transition to the Schrödinger equation are discussed. The exact solution of the Dirac equation for the particle motion in the Coulomb field with a linear dependence of the function f on the distance r to the force centre and the inverse dependence on r for the function f_1 has been found.

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Introduction

The problems with deformed Heisenberg algebra with small additions to the canonical commutation relations have been under a thorough and versatile scrutiny for a period of time [1–10]. Deformed commutation relations were studied for the first time in [11] where this issue was raised in connection with the idea of quantization of space. The question of deformation of the Heisenberg algebra can be approached along purely practical lines when solving eigenvalue problems. When we have a Hamiltonian in the Schrödinger equation with the potential not allowing us to find the exact analytical solution of the problem we can reduce it to a familiar form (for instance to the Hamiltonian of a harmonic oscillator) using generalized coordinates and momenta that fail to satisfy the Heisenberg algebra. The permutation relations between these operators are the so-called deformed relations. With this procedure we transfer the 'inconvenient' form of the Hamiltonian into a deformation of Heisenberg algebra. Sometimes this procedure makes it possible to more effectively find the approximate solutions of the Schrödinger equation. Some deforming functions allow us to treat this kind of transfer of inconveniences from the Hamiltonian onto the permutation relations as a problem in which the particle mass is position-dependent.

We can start from the beginning with a ‘good’ Hamiltonian having deformed Heisenberg algebra with a certain arbitrary deforming function dependent on both the coordinates and the momentum. Generally speaking, we will not always be able to make the inverse transition, i.e. to ‘toss’ this deformation back to the Hamiltonian. For this matter such problems are of interest in themselves, similarly to those about the motion of a particle with a position-dependent mass. At the same time we have to deal with the problem of mutual ordering of the momentum operators and the inverse mass in the kinetic energy. This problem, however, does not appear when we resort to the Dirac equation.

Thus, we arrive at the possibility of formulating the problem about the motion of the relativistic particle with a position-dependent mass in the space with deformed Heisenberg algebra. To study this problem is the aim of this paper. We also give the exact solution of the Dirac equation for the motion of a particle in the Coulomb field when its mass and deforming function are specifically dependent on the coordinates. A preliminary report on these results was given in [12].

1. The initial equations

Let us start from the Dirac equation for a particle with the potential energy U in conventional notation:

$$[(\hat{\boldsymbol{\alpha}}\hat{\mathbf{P}})c + m^*c^2\hat{\beta} + U]\Psi = E\Psi, \tag{1.1}$$

where $\hat{\boldsymbol{\alpha}}, \hat{\beta}$ are the Dirac matrices, the coordinates and momenta satisfy the permutation relations with deformed Heisenberg algebra:

$$\begin{cases} [x_j, x_k] = 0, \\ [x_j, \hat{P}_k] = \mathrm{i}\hbar\delta_{jk} f, \\ [\hat{P}_j, \hat{P}_k] = -\mathrm{i}\hbar\left(\dfrac{\partial f}{\partial x_j}\hat{P}_k - \dfrac{\partial f}{\partial x_k}\hat{P}_j\right), \qquad (j,k) = 1,2,3; \end{cases} \tag{1.2}$$

with the deforming function $f = f(x, y, z)$ dependent on the particle coordinates only. We assume that the particle mass m substituted for a certain effective mass m^* is also position-dependent:

$$m^* = mf_1, \qquad f_1 = f_1(x, y, z). \tag{1.3}$$

The embedding in the Dirac equation of the functions f and f_1 implies involvement of extra forces acting on the particle alongside those represented by the function U. We introduce a new momentum:

$$\hat{\mathbf{p}} = f^{-1/2}\hat{\mathbf{P}} f^{-1/2}, \qquad \hat{\mathbf{P}} = f^{1/2}\hat{\mathbf{p}} f^{1/2}, \tag{1.4}$$

in such a way that coordinates and new momenta become canonically conjugated

$$\begin{cases} [x_j, x_k] = 0, \\ [x_j, \hat{p}_k] = \mathrm{i}\hbar\delta_{jk}, \\ [\hat{p}_j, \hat{p}_k] = 0. \end{cases} \tag{1.5}$$

Now the Dirac equation (1.1) looks as follows:

$$[f^{1/2}(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})f^{1/2}c + mc^2 f_1\hat{\beta} + U]\Psi = E\Psi. \tag{1.6}$$

We make the transformation

$$\bar{\Psi} = f^{1/2}\Psi, \tag{1.7}$$

as a result of which equation (1.6) for the new function $\bar{\Psi}$ will be

$$[f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})c + mc^2 f_1\hat{\beta} + U]\bar{\Psi} = E\bar{\Psi}. \tag{1.8}$$

We can treat this equation as the usual Dirac equation in which the Dirac matrices $\hat{\boldsymbol{\alpha}}$ are multiplied by certain position-dependent factors:

$$\hat{\boldsymbol{\alpha}}' = f\hat{\boldsymbol{\alpha}}, \qquad \hat{\beta}' = f_1\hat{\beta}. \tag{1.9}$$

The matrix components $\hat{\boldsymbol{\alpha}}'$ and the matrix $\hat{\beta}'$ are mutually anticommuting. The squares of the components of the matrix $\hat{\boldsymbol{\alpha}}'$ equal f^2, and the square of $\hat{\beta}'$ equals f_1^2.

Before we consider the exact solutions of equation (1.8) it is expedient to pass to the nonrelativistic limit in the Dirac equation in order to find the properties of the functions f and f_1.

2. The nonrelativistic limit: the Schrödinger equation

In order to obtain the Schrödinger equation from equation (1.8) at $c \to \infty$ we introduce the new function ψ by the following relation:

$$\bar{\Psi} = [f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})c + mc^2 f_1\hat{\beta} + E - U]\psi. \tag{2.1}$$

After substituting (2.1) into (1.8) we find

$$\left\{\frac{f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})}{2m} + \frac{m^2c^4 f_1^2 - (E-U)^2}{2mc^2} + \frac{\mathrm{i}\hbar f(\hat{\boldsymbol{\alpha}}\nabla U)}{2mc} + \frac{\mathrm{i}\hbar c f}{2}\hat{\beta}(\hat{\boldsymbol{\alpha}}\nabla f_1)\right\}\psi = 0.$$

We measure energy from the rest energy mc^2,

$$E' = E - mc^2,$$

and after simple transformations we obtain

$$\left\{\frac{f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})}{2m} + U - \frac{(E'-U)^2}{2mc^2} + \frac{\mathrm{i}\hbar f(\hat{\boldsymbol{\alpha}}\hat{\nabla}U)}{2mc} + \frac{mc^2}{2}\left(f_1^2 - 1\right) + \frac{\mathrm{i}\hbar c f}{2}\hat{\beta}\,(\hat{\boldsymbol{\alpha}}\nabla f_1)\right\}\psi = E'\psi. \tag{2.2}$$

From the latter two terms in the parentheses of equation (2.2) follows the condition on the behaviour of the function f_1 in the nonrelativistic limit. Indeed, for the light velocity c to drop out of equation (2.2) when $c \to \infty$ it is necessary that $f_1^2 - 1 \sim 1/c^2$. The function f_1 can lead to one at $c \to \infty$ also faster than $1/c^2$ leaving no contribution whatsoever in the nonrelativistic limit. If

$$f_1^2 - 1 = \frac{2}{mc^2}U_1, \qquad c \to \infty, \tag{2.3}$$

where $U_1 = U_1(x, y, z)$ is a certain function of the coordinates; then from equation (2.2) we find its nonrelativistic limit:

$$\left[\frac{f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})f(\hat{\boldsymbol{\alpha}}\hat{\mathbf{p}})}{2m} + U + U_1\right]\psi = E'\psi.$$

We substitute

$$\psi = \sqrt{f}\varphi,$$

and assuming that the function f depends on the length r of the radius vector $\mathbf{r}$ after simple transformations using the properties of the matrix $\hat{\boldsymbol{\alpha}}$ we obtain the following equation:

$$\left\{\frac{(f^{1/2}\hat{\mathbf{p}}f^{1/2})^2}{2m} + U + \Delta U + U_1\right\}\varphi = E'\varphi, \tag{2.4}$$

$$\Delta U = \frac{f}{mr}\frac{\mathrm{d}f}{\mathrm{d}r}(\hat{\mathbf{S}}\hat{\mathbf{L}}), \tag{2.5}$$

where $\hat{\mathbf{S}} = \hbar\hat{\boldsymbol{\sigma}}/2$ is the operator of the particle spin, $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the Pauli matrices, $\hat{\mathbf{L}}$ is the angular momentum.

Expression (2.4) can be treated as the Schrödinger equation for a particle with the position-dependent mass $\bar{m} = m/f^2$ where the momentum operator and the inverse mass in the kinetic energy operator are specifically ordered:

$$\hat{T} = \frac{1}{2\bar{m}^{1/4}}\hat{\mathbf{p}}\frac{1}{\sqrt{\bar{m}}}\hat{\mathbf{p}}\frac{1}{\bar{m}^{1/4}}. \tag{2.6}$$

Unlike the 'standard' spin–orbital interaction, expression (2.5) for ΔU does not vanish in the nonrelativistic limit. We refer to it as the spin–orbital deformation interaction.

If we write equation (2.4) using the 'old' momentum (1.4) we have the Schrödinger equation in the space with deformed Heisenberg algebra:

$$\left(\frac{\hat{\mathbf{P}}^2}{2m} + U + \Delta U + U_1\right)\varphi = E'\varphi. \tag{2.7}$$

Hence, if in the nonrelativistic theory we start from the standard Schrödinger equation for the study of the behaviour of the particle with the deformed permutative relations (1.2), the contribution from the spin–orbital interaction ΔU gets lost as well as the term U_1 caused by the dependence of the particle mass on the coordinates (1.3).

3. The Dirac radial equation

We consider the particle motion in the central symmetrical field U and functions f, f_1 to be dependent on the distance r only. We return to the Dirac equation (1.8) and reduce it to the radial equation. In order to do so we introduce the radial momentum operator,

$$\hat{p}_r = r^{-1}(\mathbf{r}\hat{\mathbf{p}} - \mathrm{i}\hbar) \tag{3.1}$$

and a radial component of the matrix $\hat{\boldsymbol{\alpha}}$,

$$\hat{\alpha}_r = (\hat{\boldsymbol{\alpha}}\hat{\mathbf{n}}), \qquad \mathbf{n} = \frac{\mathbf{r}}{r}. \tag{3.2}$$

Further, following [13], we use the operator introduced for the first time by Dirac

$$\hbar\hat{K} = \hat{\beta}[(\hat{\boldsymbol{\sigma}}'\hat{\mathbf{L}}) + \hbar], \qquad \hat{\boldsymbol{\sigma}}' = \begin{pmatrix}\hat{\boldsymbol{\sigma}} & 0\\ 0 & \hat{\boldsymbol{\sigma}}\end{pmatrix}, \tag{3.3}$$

and calculating the product $\hat{\alpha}_r\hat{K}$ we transform equation (1.8) into the following:

$$\left(f\hat{\alpha}_r\hat{p}_r c + \frac{\mathrm{i}\hbar c f}{r}\hat{\alpha}_r\hat{\beta}\hat{K} + mc^2 f_1\hat{\beta} + U\right)\bar{\Psi} = E\bar{\Psi}. \tag{3.4}$$

The operator $\hat{K}$ is the motion integral with the eigenvalues

$$k = \pm\left(j + \frac{1}{2}\right) = \pm 1, \pm 2, \ldots, \tag{3.5}$$

j is the quantum number of the total angular momentum. That is why in the representation where the operator $\hat{K}$ is diagonal the Dirac radial equation has the form:

$$\left(f\hat{\alpha}_r\hat{p}_r c + \frac{\mathrm{i}\hbar c f}{r}\hat{\alpha}_r\hat{\beta}k + mc^2 f_1\hat{\beta} + U - E\right)\bar{R} = 0, \tag{3.6}$$

and

$$\bar{\Psi} = Y\bar{R}, \tag{3.7}$$

Y is the spherical spinor, that is the eigenvalue of the operator $\hat{K}$, $\bar{R}$ is the radial function. Now we introduce a new radial function R with the following relation (see also in [14]):

$$\bar{R} = \left(f\hat{\alpha}_r\hat{p}_r c + \frac{\mathrm{i}\hbar f}{r}\hat{\alpha}_r\hat{\beta}k + mc^2 f_1\hat{\beta} + E - U\right)R. \tag{3.8}$$

Substituting this expression into the previous equation (3.6), we find the equation for R:

$$\left\{c^2(f\hat{p}_r)^2 + \hbar^2c^2kf\hat{\beta}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f}{r}\right) + m^2c^4f_1^2 + \frac{\hbar^2c^2f^2k^2}{r^2}\right.$$
$$\left. + \mathrm{i}\hbar c f\hat{\alpha}_r\frac{\mathrm{d}U}{\mathrm{d}r} - \mathrm{i}\hbar mc^3\hat{\alpha}_r\hat{\beta}f\frac{\mathrm{d}f_1}{\mathrm{d}r} - (E-U)^2\right\}R = 0.$$

In order to separate the space variables and those describing the internal degrees of freedom we demand that the factors at the matrices $\hat{\beta}$, $\hat{\alpha}_r$, and $\hat{\alpha}_r\hat{\beta}$ have the same dependence on the variable r, i.e.,

$$C_1\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f}{r}\right) = \frac{\mathrm{d}U}{\mathrm{d}r}, \qquad C_2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f}{r}\right) = \frac{\mathrm{d}f_1}{\mathrm{d}r}, \tag{3.10}$$

where C_1, C_2 are constants.

If (3.10) holds then equation (3.9) has the form:

$$\left\{c^2(f\hat{p}_r)^2 + \hbar^2c^2\hat{\Lambda}f\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f}{r}\right) + \frac{\hbar^2c^2f^2k^2}{r^2} + m^2c^4f_1^2 - (E-U)^2\right\}R = 0, \tag{3.11}$$

where the operator

$$\hat{\Lambda} = k\hat{\beta} + \frac{\mathrm{i}}{\hbar c}\hat{\alpha}_r C_1 - \mathrm{i}\frac{mc}{\hbar}\hat{\alpha}_r\hat{\beta}C_2. \tag{3.12}$$

The operator $\hat{\Lambda}$ does not depend on the radial coordinate and it can easily be reduced to the diagonal form with eigenvalues

$$\lambda = \pm\sqrt{k^2 + \left(\frac{mc}{\hbar}C_2\right)^2 - \left(\frac{C_1}{\hbar c}\right)^2}.$$

If one works in the representation where the operator $\hat{\Lambda}$ is diagonal, our radial equation (3.11) finally gets the following form:

$$\left\{c^2(f\hat{p}_r)^2 + \hbar^2c^2\lambda f\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f}{r}\right) + \frac{\hbar^2c^2f^2k^2}{r^2} + m^2c^4f_1^2 - (E-U)^2\right\}R = 0. \tag{3.13}$$

Let us remark that as the functions f, f_1 and U are related by two conditions (3.10), only one of them is independent; for instance, it could be the potential energy U.

4. The Kepler problem

Now we consider the Kepler problem, that is the motion of the charged particle in the Coulomb field when the potential energy

$$U = -\frac{e^2}{r}, \tag{4.1}$$

where e^2 is the charge squared. From equation (3.10) we find the deforming function

$$f = 1 + \nu r, \qquad \nu > 0, \tag{4.2}$$

where ν is a constant and the function

$$f_1 = 1 + \frac{a}{r}, \tag{4.3}$$

a is a constant and

$$C_1 = -e^2, \qquad C_2 = a.$$

Then the eigenvalues of $\hat{\Lambda}$ are

$$\lambda = \pm\sqrt{k^2 + \left(\frac{mca}{\hbar}\right)^2 - \left(\frac{e^2}{\hbar c}\right)^2}. \tag{4.4}$$

After a standard substitution

$$R = \frac{\chi}{r}, \tag{4.5}$$

where $\chi = \chi(r)$, the radial equation (3.13) becomes

$$\left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\hbar^2}{2mr^2}l^*(l^*+1) - \frac{e^{*2}}{r}\right\}\chi = E^*\chi, \tag{4.6}$$

where

$$\mathrm{d}x = \frac{\mathrm{d}r}{f}.$$

From the latter we have

$$x\nu = \ln(1+\nu r), \qquad 0 \leqslant x < \infty. \tag{4.7}$$

The values with the asterisk in equation (4.6) are as follows:

$$\begin{cases} l^*(l^*+1) = k^2 + \left(\dfrac{mca}{\hbar}\right)^2 - \lambda - \left(\dfrac{e^2}{\hbar c}\right)^2, \\ e^{*2} = \dfrac{E}{mc^2}e^2 - \dfrac{\hbar^2 k^2 \nu}{m} + \dfrac{\hbar^2 \nu}{2m}\lambda - mc^2 a, \\ E^* = \dfrac{E^2 - m^2c^4}{2mc^2} - \dfrac{\hbar^2 k^2 \nu^2}{2m}. \end{cases} \tag{4.8}$$

The effective orbital quantum number

$$l^* = \begin{cases} \sqrt{k^2 - \bar{\alpha}^2} - 1 \\ \sqrt{k^2 - \bar{\alpha}^2} \end{cases}, \qquad \bar{\alpha}^2 = \alpha^2 - \left(\frac{mca}{\hbar}\right)^2, \tag{4.9}$$

$\alpha = e^2/\hbar c$ is the fine structure constant; here the upper value of l^* determines the upper sign for λ (4.4) and the lower value sets the lower sign, respectively.

Thus, equation (4.6) is split into two independent equations for the positive and negative values of the quantity λ from (4.4). If we write the radial coordinate r from equation (4.7) explicitly through x and substitute r into equation (4.6) then after simple transformations we arrive at the following equations:

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{A(A-\nu/2)}{\sinh^2(x\nu/2)}-\frac{2B}{\tanh(x\nu/2)}\right\}\chi=\varepsilon\chi, \tag{4.10}$$

where

$$\begin{aligned} A(A-\nu/2)&=\nu^2\frac{l^*(l^*+1)}{4},\\ B&=\frac{me^{*2}\nu}{2\hbar^2}+\nu^2\frac{l^*(l^*+1)}{4},\\ \varepsilon&=\frac{2m}{\hbar^2}\left[E^*-\frac{\hbar^2\nu^2 l^*(l^*+1)}{4m}-\frac{e^{*2}\nu}{2}\right]. \end{aligned} \tag{4.11}$$

It is well known that this equation has the exact solution [15] with the energy levels

$$\varepsilon=-\left(A+\frac{\nu}{2}n_r\right)^2-\frac{B^2}{(A+\nu n_r/2)^2}, \tag{4.12}$$

$n_r=0,1,2,\ldots$ is the radial quantum number and bound states exist if

$$B>A^2,\qquad A\geqslant 0,\qquad B\geqslant 0. \tag{4.13}$$

As in our case

$$A=\frac{\nu}{2}(l^*+1),$$

then from (4.12) taking into account the notation in (4.8) for the energy levels E we find the following equation:

$$\begin{aligned}\frac{E^2-m^2c^4}{mc^2}&=\frac{\hbar^2\nu^2}{2m}(k^2-\bar{\alpha}^2)-\frac{\hbar^2\nu^2}{4m}n^2\\ &\quad+\nu e^2\frac{E}{mc^2}-\nu amc^2-\frac{m}{\hbar^2n^2}\left[\frac{e^2E}{mc^2}-mc^2a-\frac{\hbar^2\nu}{2m}(k^2+\bar{\alpha}^2)\right]^2,\end{aligned} \tag{4.14}$$

$n=n_r+l^*+1$ is the principal quantum number.

It is significant that the quantity λ drops out of this equation and a dependence on this quantity remains only in the effective orbital quantum number l^*. Thus, one solution of equation (4.6) yields the radial function χ_{n_r,l^*} for $l^*=\sqrt{k^2-\bar{\alpha}^2}-1$ with the energy $E=E_{n,k}$; we have the second solution for the negative sign of the quantity λ in (4.4), it equals the function χ_{n_r,l^*+1} with the eigenvalue of energy $E_{n+1,k}$. In the nonrelativistic case, the first solution gives $l^*=l=0,1,2,\ldots$, and the second one $l^*=l=1,2,\ldots$, where l is the usual orbital quantum number. Thus the energy levels for the two solutions coincide with the exception of the ground state. Here we have the so-called supersymmetry. The Dirac equation (1.8) for the Coulomb potential with the deforming functions f and f_1 satisfying conditions (3.10) reveals supersymmetry. But this issue calls for a separate study. It would be interesting to compare our result concerning supersymmetry with the results obtained earlier (see, for instance, [17, 18]). Solving equation (4.14) for $E=E_{n,k}$ we finally find

$$\begin{aligned}E&=\frac{\nu e^2}{2}\frac{(n^2+k^2+\bar{\alpha}^2)}{n^2+\alpha^2}+\left(\frac{mc}{\hbar}\right)^2\frac{e^2a}{n^2+\alpha^2}\\ &\quad+\frac{mc^2}{1+\alpha^2/n^2}\left\{1+\frac{\bar{\alpha}^2}{n^2}+\left(\frac{\nu e^2}{2mc^2}\right)^2\left(1+\frac{k^2+\bar{\alpha}^2}{n^2}\right)\left(1+a\nu+\frac{k^2+\bar{\alpha}^2}{n^2}\right)\right.\\ &\quad\left.+\left(\frac{\hbar\nu}{2mc}\right)^2\left(1+\frac{\alpha^2}{n^2}\right)\left[2(k^2-\bar{\alpha}^2)-n^2-\frac{(k^2+\bar{\alpha}^2)^2}{n^2}\right]\right\}^{1/2}.\end{aligned} \tag{4.15}$$

The condition for the existence of bound states follows from (4.13):

$$\frac{E}{mc^2}e^2 > \frac{\hbar^2\nu}{m}k^2 + mc^2 a. \tag{4.16}$$

The initial function Ψ contained in equation (1.1) is found from (1.7), (3.7), (3.8) and (4.5):

$$\Psi = f^{-1/2} Y \left(f\hat{\alpha}_r \hat{p}_r c + \frac{\mathrm{i}\hbar f}{r}\hat{\alpha}_r \hat{\beta} k + mc^2 f_1 \hat{\beta} + E - U \right) \frac{\chi}{r}, \tag{4.17}$$

where χ is the matrix column with the elements χ_{n_r,l^*} and χ_{n_r,l^*+1}.

Formulae (4.15)–(4.17) provide the exact solution of the Kepler problem in the Dirac theory with Heisenberg algebra that is deformed by function (4.2) with the position-dependent particle mass in accordance with (1.3), (4.3).

5. Discussion of the results

If in (4.15) we put $\nu = 0$, i.e. we remove deformation, the energy levels for the Dirac charged particle whose mass is position-dependent are obtained:

$$E = \frac{mc^2}{1 + \alpha^2/n^2} \left(\frac{me^2 a}{\hbar^2 n^2} + \sqrt{1 + \frac{\bar{\alpha}^2}{n^2}} \right), \tag{5.1}$$

and

$$a < \frac{e^2}{mc^2}$$

which follows from (4.16). This result was originally discovered in [16] and reproduced in [19] by a different technique.

The nonrelativistic limit, $c \to \infty$, for expression (4.15) was found. We assume that the function f_1 satisfies condition (2.3), otherwise, we believe that the dependence of the particle mass on its coordinates makes its own contributions to the nonrelativistic limit. It means that taking into account the explicit form of the function f_1 (4.3) the parameter $a \sim 1/c^2$. That is why we take

$$a = \frac{e^2}{mc^2}\bar{a},$$

where $\bar{a}$ is a dimensionless constant. In this case the nonrelativistic limit for the energy E is as follows:

$$E' = E - mc^2 = -\frac{m}{2\hbar^2 n^2}\left(e^2 - \frac{\hbar^2\nu}{2m}k^2\right)^2 - \frac{\hbar^2\nu^2}{8m}n^2 + \frac{\nu}{2}\left(e^2 + \frac{\hbar^2\nu}{2m}k^2\right) + \frac{me^4}{2\hbar^2 n^2}\bar{a}(2 - \bar{a}), \tag{5.2}$$

and in accordance with (4.16) the energy spectrum is limited.

$$e^2 > \frac{\hbar^2\nu}{m}k^2 + e^2\bar{a}. \tag{5.3}$$

It is interesting to compare expressions (5.2) and (5.3) when $\bar{a} = 0$ with the results in [20] where the Schrödinger equation for a particle in the Coulomb field with the deforming function (4.2), in our notation, was solved:

$$E'_{\mathrm{QT}} = -\frac{m}{2\hbar^2 n^2}\left(e^2 - \frac{\hbar^2\nu}{2m}[l(l+1)+1]\right)^2 - \frac{\hbar^2\nu^2}{8m}n^2 + \frac{\nu}{2}\left(e^2 + \frac{\hbar^2\nu}{2m}[l(l+1)+1]\right), \tag{5.4}$$

with the condition that

$$e^2 > \frac{\hbar^2 \nu}{2m}[(l+1)(2l+1)+1].$$

The difference of this expression from formula (5.2) is explained by the fact that the authors of [20] disregarded the deformational spin–orbital interaction ΔU which arises naturally in our treatment in the nonrelativistic limit from the Dirac equation. These authors started from the Schrödinger equation. If the said interaction is not taken into account, we must deduce the contribution from

$$\Delta U = \frac{\nu^2}{m}(\hat{\mathbf{S}}\hat{\mathbf{L}}) + \frac{\nu}{m}(\hat{\mathbf{S}}\hat{\mathbf{L}})\frac{1}{r}, \tag{5.5}$$

which follows from (2.5) and (4.2). As the eigenvalue of the operator $(\hat{\mathbf{S}}\hat{\mathbf{L}}) = (\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2)/2$ equals $\hbar^2[j(j+1) - l(l+1) - 3/4]/2 = \hbar^2[(j+1/2)^2 - l(l+1) - 1]/2 = \hbar^2[k^2 - l(l+1) - 1]/2$, this contribution can easily be taken into account. Indeed, in order to remove the contribution of ΔU from our result it is necessary to deduce from the energy E' the contribution of the first term in (5.5) equaling $\hbar^2\nu^2[k^2 - l(l+1) - 1]/2m$; the second term in (5.5) should be united with the Coulomb potential (4.1) by the substitution: $e^2 \to e^2 + \hbar^2\nu[k^2 - l(l+1) - 1]/2m$. Consequently, from (5.2) we arrive at expression (5.4). Besides, we must put $j = l + 1/2$ in condition (5.3) limiting the spectrum in expression (3.5) for the quantum number k. In other words, we should take a higher value of $k^2 = (l+1)^2$.

Now we give the next after the zeroth approximation (5.2) term of the development of energy $E^{(1)}$ by the degrees $1/c^2$. We represent $E^{(1)}$ as a sum of three terms:

$$E^{(1)} = \Delta_1 E^{(1)} + \Delta_2 E^{(1)} + \Delta_3 E^{(1)}. \tag{5.6}$$

The correction does not depend on the parameter ν [19]:

$$\Delta_1 E^{(1)} = -\frac{me^4}{2\hbar^2}\frac{\alpha^2}{n^4}(1-\bar{a})^3\left[\frac{n}{|k|}(1+\bar{a}) - \frac{3}{4}(1+\bar{a}/3)\right]. \tag{5.7}$$

At $\bar{a} = 0$ it transforms into the well-known Sommerfeld formula. The correction

$$\Delta_2 E^{(1)} = -\left(\frac{\hbar\nu}{8mc}\right)^2 \frac{\hbar^2\nu^2}{2m}\frac{(n^2-k^2)^4}{n^4} \tag{5.8}$$

is brought about only by deformation. The cross term

$$\begin{aligned}\Delta_3 E^{(1)} = {} & \frac{\nu e^2}{2}\frac{\alpha^2}{n^4}[(1-\bar{a}^2)n|k| - k^2 - n^2\bar{a}^2] + \frac{\hbar^2\nu^2}{8m}\frac{\alpha^2}{n^4} \\ & \times\left\{(n^2+k^2)^2 + (1-\bar{a}^2)\left[2k^4 - \frac{3}{2}(n^2+k^2)^2 + \frac{n}{|k|}(n^4-k^4)\right]\right\}.\end{aligned} \tag{5.9}$$

The obtained results are of general interest. They can also be useful for the study of the energy spectrum of nanoheterosystems when the electron mass is position dependent and also whenever it is important to take into account relativistic effects, in particular those of spin–orbital interaction.

Finally, let us mention that the question of the application of deformed commutation relations in the Kepler relativistic problem remains open. As the non-deformed Kepler problem is Lorenz-invariant the question arises whether this property will be preserved in the deformed space. Though it is well known that the quantum spacetime with deformed Heisenberg algebra can be Lorenz-invariant [11], it is obvious that in our case the problem is not like that. A similar controversy was found in [21] that studied the Dirac oscillator with deformed commutation relations leading to the existence of the minimal length of space. However, this problem calls for a more detailed investigation to be suggested in my next paper.

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